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Trigonometric Functions in the m-plane

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ABSTRACT

In this paper, we define the trigonometric functions in the plane with the m-metric. And then we give two properties about these trigonometric functions, one of which states the area formula for a triangle in the m-plane in terms of the m-metric.

Key words: Taxicab metric, Chinese checker metric, alpha metric, m-metric, m-trigonometry

MSC2010: 51K05, 51K99

Trigonometrijske funkcije u m-ravnini

SAŽETAK

U članku definiramo trigonometrijske funkcije u ravnini s m-metrikom. Zatim pokazujemo dva svojstva ovih trigonometrijskih funkcija gdje jedno od njih daje formulu površine trokuta u m-ravnini s primjenom m-metrike.

Ključne riječi: Taxicab metrika, metrika kineskog šaha, alfa metrika, m-metrika, m-trigonometrija

1 Introduction

The *taxicab metric* was given in a family of metrics of the real plane by Minkowski [16]. And the taxicab geometry introduced by Menger [15], and developed by Krause [14]. Later, Chen [7] developed the *Chinese checker metric*, and Tian [19] gave a family of metrics, α-metric for $\alpha \in [0,\pi/4]$, which includes the taxicab and Chinese checker metrics as special cases, and Çolakoğlu [8] extended the α -metric for $\alpha \in [0,\pi/2)$. Afterwards, Bayar, Ekmekçi and Akça [5] presented a generalization of α-metric: the *generalized absolute value metric*. Finally, Çolakoğlu and Kaya [10] gave a generalization for all these metrics: m-metric (or m-generalized absolute value metric). During the recent years, trigonometry on the plane geometries based on these metrics have been studied. See [1], [2], [3], [4], [5], [6], [12], [17] and [18] for some of studies. In this paper, we study on trigonometry in the plane with the generalized m-metric. First, we give definitions of trigonometric functions for the m-metric, which also generalize the definitions given before, and then give two properties about these trigonometric functions, one of which states a formula to calculate the area of any triangle in the m-plane, being an alternative to the one given in [13]. This study also provides a facility for further subjects as cosine theorem, norm and inner-product in terms of the m-metric.

Let $P_1=(x_1,y_1)$ and $P_2=(x_2,y_2)$ be two points in $\mathbb{R}^2$. For each real numbers a and b, such that $a \geq b \geq 0 \neq a$, the function $d_m : \mathbb{R}^2 \times \mathbb{R}^2 \to [0,\infty)$ defined by

$$d_m(P_1,P_2) = (a\Delta_{AB} + b\delta_{AB}) \big/ \sqrt{1+m^2} \tag{1}$$

where

$$\Delta_{AB}=\max\{|(x_1-x_2)+m(y_1-y_2)|,|m(x_1-x_2)-(y_1-y_2)|\}$$

and

$$\delta_{AB}=\min\{|(x_1-x_2)+m(y_1-y_2)|,|m(x_1-x_2)-(y_1-y_2)|\},$$

is called the m- *distance function* in $\mathbb{R}^2$, and the real number $d_m(P_1,P_2)$ is called the *m-distance* between points P_1 and P_2.

Cartesian coordinate plane endowed with the m-metric forms a metric space, $\mathbb{R}^2_m$ or $(\mathbb{R}^2, d_m)$, and it is constructed by simply replacing the well-known Euclidean distance function

$$d_E(P_1,P_2) = ((x_1-x_2)^2 + (y_1-y_2)^2)^{1/2} \tag{2}$$

by the m-distance function d_m in $\mathbb{R}^2$ (see [10]). In all that follows, we use $a' = a\big/\sqrt{1+m^2}$ and $b' = b\big/\sqrt{1+m^2}$ to shorten phrases.

2 Trigonometric Functions

We know that if $P = (x,y)$ is a point on the Euclidean unit circle, then $x = \cos\theta$ and $y = \sin\theta$, where θ is the angle with the positive x-axis as the initial side and the radial line passing through the point P as the terminal side. One can determine the standard definitions of the trigonometric functions using the unit m-circle in $\mathbb{R}^2_m$, in the same way one determines their Euclidean analogues. The unit m-circle (see Figure 1) is the set of points (x,y), which satisfies the equation

$$a' \max\{|x+my|,|mx-y|\}+b' \min\{|x+my|,|mx-y|\}=1. \tag{3}$$

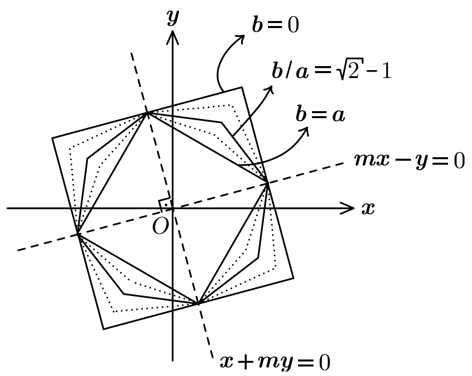

Figure 1: *Graph of unit m-circles*

So, for the point $P = (x,y)$ on the m-unit circle, let us determine *sine* and *cosine* functions in $\mathbb{R}^2_m$ as $x = \cos_m\theta$ and $y = \sin_m\theta$, where θ is the angle with the positive x-axis as the initial side and the radial line passing through the point P as the terminal side. Clearly, *tangent* and *cotangent* functions do not depend on the metric, since the slope of the radial line passing through the point (x,y) does not change. Thus, we have

$$\tan_m\theta = \frac{\sin_m\theta}{\cos_m\theta} = \tan\theta \quad \text{and} \quad \cot_m\theta = \frac{\cos_m\theta}{\sin_m\theta} = \cot\theta.$$

Obviously, the equation of the line joining (x,y) and $(0,0)$ is $y = (\tan\theta)x$. Solving the system

$$\begin{cases} y = (\tan\theta)x \\ a' \max\{|x+my|,|mx-y|\}+b' \min\{|x+my|,|mx-y|\}=1 \end{cases}$$

one gets *sine* and *cosine* functions in $\mathbb{R}^2_m$:

$$\cos_m\theta = \frac{\cos\theta}{a' \max\{X,Y\}+b' \min\{X,Y\}}, \tag{4}$$

$$\sin_m\theta = \frac{\sin\theta}{a' \max\{X,Y\}+b' \min\{X,Y\}}, \tag{5}$$

where $X = |\cos\theta + m\sin\theta|$, $Y = |m\cos\theta - \sin\theta|$.

We can also determine *secant* and *cosecant* functions as in Euclidean plane: $\csc_m\theta = \frac{1}{\sin_m\theta}$ and $\sec_m\theta = \frac{1}{\cos_m\theta}$. For some values of a, b and m, graphs of $y = \sin_m x$ and $y = \cos_m x$ are given in Figure 2, Figure 3, Figure 4 and Figure 5, for $-2\pi < x < 2\pi$.

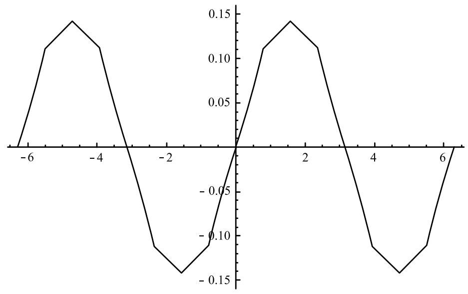

Figure 2: *Graph of* $y = \sin_m x$ *for* $a=7$, $b=2$ *and* $m=0$

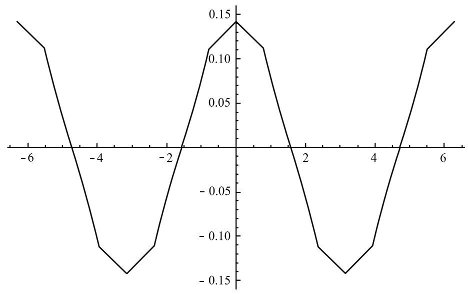

Figure 3: *Graph of* $y = \cos_m x$ *for* $a=7$, $b=2$ *and* $m=0$

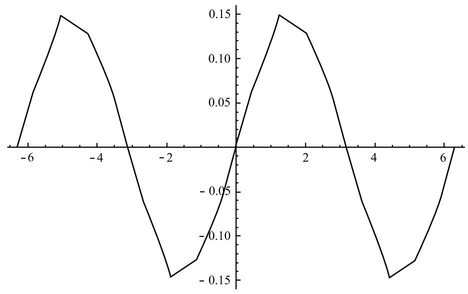

Figure 4: *Graph of* $y = \sin_m x$ *for* $a=7$, $b=2$ *and* $m=\frac{1}{2}$

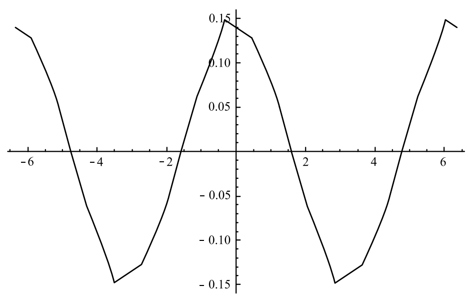

Figure 5: *Graph of* $y = \cos_m x$ *for* $a=7$, $b=2$ *and* $m=\frac{1}{2}$

In $\mathbb{R}^2_m$, the trigonometric identities differ from their Euclidean analogues in most cases. Some of the identities of these functions are like their Euclidean counterparts:

$$\cos_m(\frac{\pi}{2}+\theta) = -\sin_m\theta, \quad \sin_m(\frac{\pi}{2}+\theta) = \cos_m\theta$$
$$\cos_m(\frac{3\pi}{2}+\theta) = \sin_m\theta, \quad \sin_m(\frac{3\pi}{2}+\theta) = -\cos_m\theta$$
$$\cos_m(\pi+\theta) = -\cos_m\theta, \quad \sin_m(\pi+\theta) = -\sin_m\theta$$
$$\cos_m(2\pi+\theta) = \cos_m\theta, \quad \sin_m(2\pi+\theta) = \sin_m\theta.$$

It is well known that the Pythagorean identity is the relation between sine and cosine functions: $\sin^2\theta+\cos^2\theta = 1$. In terms of the generalized m-metric, we get the following equation

$$a'\max\{|\cos_m\theta + m\sin_m\theta|, |m\cos_m\theta - \sin_m\theta|\} \\ +b'\min\{|\cos_m\theta + m\sin_m\theta|, |m\cos_m\theta - \sin_m\theta|\} = 1. \tag{6}$$

One can also get the following equations easily:

$$\begin{aligned} &a'\max\{|1+m\tan\theta|, |m-\tan\theta|\} \\ &\quad +b'\min\{|1+m\tan\theta|, |m-\tan\theta|\} = |\sec_m\theta| \\ &a'\max\{|\cot\theta+m|, |m\cot\theta-1|\} \\ &\quad +b'\min\{|\cot\theta+m|, |m\cot\theta-1|\} = |\csc_m\theta|. \end{aligned} \tag{7}$$

Using the sum and difference formulas for tangent function one gets also the following equations:

$$\begin{aligned} \tan_m(u\mp v) &= \frac{\tan_m u \mp \tan_m v}{1\pm\tan_m u\tan_m v} \\ \cot_m(u\mp v) &= \frac{1\pm\cot_m u\cot_m v}{\cot_m u\mp\cot_m v} \end{aligned} \tag{8}$$

$$\begin{aligned} \sin_m(u\mp v) &= \sin_m u\cos_m v \mp \sin_m v\cos_m u \\ \cos_m(u\mp v) &= \cos_m u\cos_m v \pm \sin_m u\sin_m v. \end{aligned} \tag{9}$$

3 Trigonometric Functions with Reference Angle

Unlike the Euclidean case, there is a non-uniform increment in the arc length as the angle θ is incremented by a fix amount, in $\mathbb{R}^2_m$. So, it is necessary to develop the trigonometric functions for any angle θ using the reference angle α of θ (see [18]).

Definition 1 *Let θ be an angle with the reference angle α which is the angle between θ and the positive direction of the x-axis in m -unit circle. Then the cosine and sine functions of the angle θ with the reference angle α,* $\mathrm{m}\cos\theta$ *and* $\mathrm{m}\sin\theta$, *are defined by*

$$\mathrm{m}\cos\theta = \cos_m(\alpha+\theta)\cos_m\alpha + \sin_m(\alpha+\theta)\sin_m\alpha \tag{10}$$

$$\mathrm{m}\sin\theta = \sin_m(\alpha+\theta)\cos_m\alpha - \cos_m(\alpha+\theta)\sin_m\alpha. \tag{11}$$

In this definition, the angles of α and $(\alpha+\theta)$ are in standard position. So, the values of $\cos_m(\alpha+\theta)$, $\sin_m(\alpha+\theta)$, $\cos_m\alpha$ and $\sin_m\alpha$ are calculated by using equations (4) and (5). If $\alpha = 0$, then

$$\mathrm{m}\cos\theta = \frac{\cos_m\theta}{a'\max\{1,|m|\}+b'\min\{1,|m|\}} \tag{12}$$

$$\mathrm{m}\sin\theta = \frac{\sin_m\theta}{a'\max\{1,|m|\}+b'\min\{1,|m|\}}. \tag{13}$$

The general definitions of other trigonometric functions for the angles which are not in standard position can be given similarly: $\mathrm{m}\tan\theta = \frac{\mathrm{m}\sin\theta}{\mathrm{m}\cos\theta} = \tan\theta$, $\mathrm{m}\cot\theta = \frac{\mathrm{m}\cos\theta}{\mathrm{m}\sin\theta} = \cot\theta$, $\mathrm{m}\csc\theta = \frac{1}{\mathrm{m}\sin\theta}$ and $\mathrm{m}\sec\theta = \frac{1}{\mathrm{m}\cos\theta}$. Consequently, the general definitions of trigonometric functions can be given by defining angles with the reference angle in plane with the generalized m-metric.

It is well-known that all rotations and translations preserve the Euclidean distance. In $\mathbb{R}^2_m$, all translations and the rotations of the angle $\theta\in\{\frac{\pi}{2},\pi,\frac{3\pi}{2},2\pi\}$ when $b/a\neq\sqrt{2}-1$ and also the rotations of the angle $\theta\in\{\frac{\pi}{4},\frac{3\pi}{4},\frac{5\pi}{4},\frac{7\pi}{4}\}$ when $b/a=\sqrt{2}-1$ preserve m-distance (see [10]). The change of the length of a line segment by a rotation can be given by the following theorem:

Theorem 1 *Let any two points be A and B in $\mathbb{R}^2_m$, and let the line segment AB be not parallel to the x-axis and the angle α between the line segment AB and the positive direction of x-axis. If $A'B'$ is the image of AB under the rotation with the angle θ, then*

$$d_m(A',B') = d_m(A,B)\sqrt{\frac{\cos^2_m\alpha+\sin^2_m\alpha}{\cos^2_m(\alpha+\theta)+\sin^2_m(\alpha+\theta)}} \tag{14}$$

Proof. Since all translations preserve the m-distance, the line segment AB can be translated to the line segment OX such that O is the origin. Let the line segment OX' be the image of OX under rotation with the angle θ, and let $d_m(A,B) = d_m(O,X) = k$ and $d_m(O,X') = k'$. If α is the reference angle of θ, then $X = (k\cos_m\alpha, k\sin_m\alpha)$ and $X' = (k'\cos_m(\alpha+\theta), k'\sin_m(\alpha+\theta))$. Since $d_E(O,X) = d_m(O,X')$, one gets

$$k'\sqrt{\cos^2_m(\alpha+\theta)+\sin^2_m(\alpha+\theta)} = k\sqrt{\cos^2_m\alpha+\sin^2_m\alpha}$$

and

$$d_m(A',B') = d_m(A,B)\sqrt{\frac{\cos^2_m\alpha+\sin^2_m\alpha}{\cos^2_m(\alpha+\theta)+\sin^2_m(\alpha+\theta)}}. \qquad \square$$

The following corollary shows how one can find the generalized m-length of a line segment, after a rotation with an angle θ in standard position:

Corollary 1 *If the line segment AB is parallel to the x-axis, then*

$$d_m(A',B') = \frac{d_m(A,B)}{(a'\max\{1,|m|\}+b'\min\{1,|m|\})\sqrt{\cos_m^2\theta+\sin_m^2\theta}} \tag{15}$$

Proof. Since $\alpha = 0$, proof is obvious. □

In [13], an area formula for a triangle is given in the plane with the generalized m-metric (see also [11]). In the following theorem, the area of a triangle is given by using the trigonometric functions in $\mathbb{R}_m^2$.

Theorem 2 *Let ABC be any triangle in $\mathbb{R}_m^2$, and let θ be the angle between the line segments AC and BC. Then the area $\mathcal{A}$ of the triangle ABC can be given by the following formula:*

$$\mathcal{A} = \frac{1}{2}d_m(A,C)d_m(B,C)\mathrm{m}\sin\theta. \tag{16}$$

Proof. Let $d_m(A,C) = k$ and $d_m(B,C) = k'$. We can take the vertex C as the origin, and $A = (k\cos_m\alpha, k\sin_m\alpha)$ and $B = (k'\cos_m(\alpha+\theta), k'\sin_m(\alpha+\theta))$, without loss of generality. Thus, we have $d_E(A,C) = k\sqrt{\cos_m^2\alpha+\sin_m^2\alpha}$ and $d_E(B,C) = k'\sqrt{\cos_m^2(\alpha+\theta)+\sin_m^2(\alpha+\theta)}$. Also, it is easy to show that if γ is in standard position, then $\cos_m\gamma = \cos\gamma\sqrt{\cos_m^2\gamma+\sin_m^2\gamma}$ and $\sin_m\gamma = \sin\gamma\sqrt{\cos_m^2\gamma+\sin_m^2\gamma}$. Thus, one gets the equation

$$\mathrm{m}\sin\theta = \sin\theta\sqrt{\cos_m^2\alpha+\sin_m^2\alpha}\sqrt{\cos_m^2(\alpha+\theta)+\sin_m^2(\alpha+\theta)}. \tag{17}$$

If we use the values of $d_E(A,C)$, $d_E(B,C)$ and $\sin\theta$ in the formula $\mathcal{A} = \frac{1}{2}d_E(A,C)d_E(B,C)\sin\theta$, we get the area formula in the plane with the generalized m-metric: $\mathcal{A} = \frac{1}{2}d_m(A,C)d_m(B,C)\mathrm{m}\sin\theta$. □

References

[1] Z. Akça, R. Kaya, On Taxicab Trigonometry, *J. Inst. Math. Comput. Sci. Math. Ser.)* **10**(**3**) (1997), 151–159.

[2] A. Bayar, S. Ekmekçi, On the Chinese-Checker Sine and Cosine Functions, *Int. J. Math. Anal.* **1**(**3**) (2006), 249–259.

[3] A. Bayar, On Trigonometric Functions in Maximum Metric, *KoG* **12** (2008), 45–48.

[4] A. Bayar, S. Ekmekçi, M. Özcan, On Trigonometric Functions and Cosine and Sine Rules in Taxicab Plane, *Int Electron. J. Geom.* **2**(**1**) (2009), 17–24.

[5] A. Bayar, S. Ekmekçi, Z. Akça, On the Plane Geometry with Generalized Absolute Value Metric, *Math. Probl. Eng.* **2008** (2008), Art. ID 673275, 8 pp.

[6] R. Brisbin, P. Artola, Taxicab Trigonometry, *Pi Mu Epsilon J.* **8**(**3**) (1985), 249–259.

[7] G. Chen, *Lines and Circles in Taxicab Geometry*, Master Thesis, Department of Mathematics and Computer Science, University of Central Missouri, 1992.

[8] H.B. Çolakoğlu, Concerning the Alpha Distance, *Algebras Groups Geom.* **28** (2011), 1–14.

[9] H.B. Çolakoğlu, Ö. Gelişgen, R. Kaya, Pythagorean Theorems in the alpha Plane, Math. Commun. **14**(**2**) (2009), 211–221.

[10] H.B. Çolakoğlu, R. Kaya, A Generalization of Some Well-Known Distances and Related Isometries, *Math. Commun.* **16**(**1**) (2011), 21–35.

[11] H.B. Çolakoğlu, Ö. Gelişgen, R. Kaya, Area Formulas for a Triangle in the Alpha Plane, *Math. Commun.* **18** (2013), 123–132.

[12] S. Ekmekçi, Z. Akça, A.K. Altintaş, On Trigonometric Functions And Norm In The Generalized Taxicab Metric, *Math. Sci. Appl. E-Notes* **3**(**2**) (2015), 27–33.

[13] Ö. Gelişgen, T. Ermiş, Area Formulas For A Triangle In The m-Plane, *Konuralp J. Math.* **2**(**2**) (2014) 85–95.

[14] E.F. Krause, *Taxicab Geometry*, Addison-Wesley, Menlo Park, California, 1975; Dover Publications, New York, 1987.

[15] K. Menger, *You Will Like Geometry*, Guidebook of Illinois Institute of Technology Geometry Exhibit, Museum of Science and Industry, Chicago, Illinois, 1952.

[16] H. Minkowski, *Gesammelte Abhandlungen*, Chelsa Publishing Co. New York, 1967.

[17] M. Özcan, S. Ekmekçi, A. Bayar, A Note on the Variation of the Taxicab Lengths Under Rotations, *Pi Mu Epsilon J.* **11**(**7**) (2002), 381–384.

[18] K. Thompson, T. Dray, Taxicab Angles and Trigonometry, *Pi Mu Epsilon J.* **11**(**2**) (2000), 87–97.

[19] S. Tian, Alpha Distance-A Generalization of Chinese Checker Distance and Taxicab Distance, *Missouri J. Math. Sci.* **17**(**1**) (2005), 35–40.

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